A GELFAND-BEURLING TYPE FORMULA FOR HEIGHTS ON ENDOMORPHISM RINGS

Valerio Talamanca

Universiteit van Amsterdam

October 4, 1999

Abstract. Let V be a finite dimensional vector space over a number field. In this paper we prove a limit formula for heights on the endomorphism ring of V, which can be considered as the analogue for both the Gelfand-Beurling formula for the spectral radius on a Banach Algebra and Tate's averaging procedure for constructing canonical heights on Abelian Varieties. We also prove a version of Northcott's finiteness theorem.

Introduction

Let A be an abelian variety defined over a number field K. Fix an ample and symmetric invertible sheaf $\mathcal{L}$ on A. Suppose that $\phi : A \to \mathbb{P}^n$ is an injective morphism associated to $\mathcal{L}$, i.e. $\phi^*\mathcal{O}_{\mathbb{P}^n}(1) \simeq \mathcal{L}$. Let H be the Northcott-Weil ℓ^2-height on $\mathbb{P}^n(\overline{K})$ (we will recall the definition of H in section 1), and set $h_\phi = \log(H \circ \phi) : A(\overline{K}) \to \mathbb{R}$. The function h_ϕ is called a (logarithmic) Northcott-Weil height on A associated to $\mathcal{L}$. Clearly h_ϕ is not uniquely determined by $\mathcal{L}$, but it can be shown (see e.g. [9]) that the various functions constructed using different choices of ϕ (always subject to the condition $\phi^*\mathcal{O}_{\mathbb{P}^n}(1) \simeq \mathcal{L}$) all lie in the same class modulo bounded functions. A lemma of J. Tate, as stated in [8, pp. 29-30], allows us to choose, amongst all the Northcott-Weil heights associated to $\mathcal{L}$, a canonical one having good functorial properties. If we denote by $[n] : A \longrightarrow A$ the multiplication by n map, then we can explicitly compute the canonical height $\hat{h}_\mathcal{L}$ via Tate's averaging procedure:

$$\hat{h}_\mathcal{L}(P) = \lim_{k\to\infty} \frac{h_\phi([n^k]P)}{n^{2k}}. \tag{0.1}$$

Let us stress that the function $\hat{h}_\mathcal{L}$ is independent of the choice of ϕ and $[n]$.

1991 *Mathematics Subject Classification*. Primary 11R54, 11C20 11G35: Secondary 16S50, 15A60.

Key words and phrases. Heights, Endomorphism rings, Gelfand-Beurling Formula.

Research partially supported by a post-doc fellowship of the University of Padova and C.N.R. The author would also like to thank the Mathematics Department of Università di Roma III for its kind hospitality.

The purpose of this paper is to prove an analogous formula for heights on the endomorphism rings of a finite dimensional K-vector space. As for the case of abelian varieties one needs some additional structure in order to be able to define heights. In this paper we work with heights associated to an adelic norm on V. We will provide all the relevant definitions in section 1. Let V be a finite dimensional K-vector space and $\mathrm{End}\big(V\big)$ its endomorphism ring. To any adelic norm $\mathcal{F}$ on V we associate an height function $H^{\mathrm{op}}_{\mathcal{F}}$ on $\mathrm{End}\big(V\big)$. On $\mathrm{End}\big(V\big)$ there also exists another height function, H_s, which is called the spectral height, and is substantially different from the operator heights being defined as the product of the local spectral radii. In our situation the spectral height will play the role of the canonical height. In fact our first main result is:

Theorem A. *Let V be a finite dimensional K-vector space, and $\mathcal{F}$ an adelic norm on V. Then, for all $T\in\mathrm{End}\big(V\big)$, we have*

$$\lim_{k\to\infty} H^{\mathrm{op}}_{\mathcal{F}}(T^k)^{\frac{1}{k}} = H_s(T). \tag{0.2}$$

Clearly (0.2) is the analogue of Tate's averaging procedure (0.1) in this setting. Recall that the Gelfand-Beurling formula for the spectral radius on a complex Banach algebra with 1, $\big(A, \|\cdot\|\big)$, states that

$$\lim_{k\to\infty} \| a^k \|^{\frac{1}{k}} = \rho(a),$$

where $\rho(a) = \sup_{\lambda\in\,\mathrm{sp}(a)} |\lambda|$ and $\mathrm{sp}(a) = \{\lambda\in\mathbb{C} \mid a - \lambda\cdot 1 \text{ is not invertible}\}$. Therefore (0.2) can also be considered as the global analogue of the (finite dimensional) Gelfand-Beurling formula. Let us remark that the Gelfand-Beurling formula, and its real and p-adic counterparts, are actually used in the proof of theorem A.

Let $\mathbb{P}\big(\mathrm{End}(K^n)\big)$ denote the projective space associated to $\mathrm{End}(K^n)$. The operator height, being homogeneous, descends to a real valued function on $\mathbb{P}\big(\mathrm{End}(K^n)\big)$. The other significant result that we will present is the extension to this setting of Northcott's theorem stating the finiteness of the set of points of bounded heights in projective spaces:

Theorem B. *Let V be a finite dimensional K-vector space and $\mathcal{F}$ a regular adelic norm on V. Then the set*

$$\big\{[T]\in\mathbb{P}\big(\mathrm{End}(V)\big) \bigm| \mathrm{rank}\,(T) \geq 2 \text{ and } H^{\mathrm{op}}_{\mathcal{F}}(T) \leq B\big\}$$

is finite for every $B \geq 1$.

It is necessary to exclude rank one transformations from the above statement, see section 4 for more details.

The paper is organized as follows: in section 1 we set our notations and we introduce the height functions that we will use. In section 2 we prove a reduction lemma which shows that is sufficient to prove our main result for the ℓ^2 height on K^n. Section 3 contains our main technical result (theorem 3.3) which is a comparison result between different heights on $\mathrm{End}\big(K^n\big)$. Section 4 is devoted to the proof of the main results.

Acknowledgements I would like to thank D. Masser and P. Stevenhagen for many comments and suggestions. It is a pleasure to express my gratitude to Alan Mayer for encouragement and advice.

1. Heights

Let K be a number field of degree d over $\mathbb{Q}$. We denote by $\mathcal{M}_K$ the set of equivalence classes of absolute values of K; by $\mathcal{M}_K^0$ (respectively $\mathcal{M}_K^\infty$) the subset of $\mathcal{M}_K$ consisting of the equivalence classes of non-archimedean (resp. archimedean) absolute values. If $v \in \mathcal{M}_K^0$, $v|p$, we normalize $|\cdot|_v$ by requiring that $|p|_v = p^{-1}$; while if $v \in \mathcal{M}_K^\infty$ we then normalize $|\cdot|_v$ by requiring that its restriction to $\mathbb{Q}$ is the standard archimedean absolute value. Let K_v be the completion of K with respect to $|\cdot|_v$. We denote by n_v the local degree, and set $d_v = n_v/d$.

Before giving the definition of adelic norm we need to recall a few facts about lattices over number field and their completions, see [13, ch.2 & 5] for more details. Given $v \in \mathcal{M}_K^0$ we denote by $\mathcal{O}_v$ the closure of $\mathcal{O}_K$ (the ring of integers of K) in K_v. A K_v-lattice in a finite dimensional K_v-vector space is a compact and open $\mathcal{O}_v$-module. Let W be a K_v-vector space and let $M \subset W$ be a K_v-lattice M. The norm associated to M, $N_M : W \to \mathbb{R}$, is defined by

$$N_M(\mathbf{x}) = \inf_{\gamma \in K_v^\times, \gamma\mathbf{x} \in M} |\gamma|_v^{-1}.$$

Let V be an n-dimensional K-vector space. An $\mathcal{O}_K$-module Λ in V is called a K-lattice if it is finitely generated and contains a basis of V over K. Given $v \in \mathcal{M}_K^0$ we denote by Λ_v the closure of Λ in K_v^n.

By an *adelic norm*[1] (cf. [14]) on V we mean a collection $\mathcal{F} = \{N_v \, , v \in \mathcal{M}_K\}$ of norms $N_v : V_v = V \otimes_K K_v \longrightarrow \mathbb{R}$, having the following properties:

(a) N_v is a norm with respect to $|\cdot|_v$. Moreover, if $v \in \mathcal{M}_K^0$, then N_v is ultrametric, i.e. $N_v(\mathbf{x}+\mathbf{y}) \leq \max\{N_v(\mathbf{x}), N_v(\mathbf{y})\}$.

(b) There exists a K-lattice Λ, such that N_v is the norm associated to Λ_v for all but finitely many $v \in \mathcal{M}_K^0$.

A moment of reflection shows that if Λ is a K-lattice in V, then there exists a basis $\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$ such that for all but finitely many $v \in \mathcal{M}_K^0$, Λ_v is the $\mathcal{O}_v$-module generated by $\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$. In particular for each $\mathbf{x} \in V$ the set $\{v \in \mathcal{M}_K^0 | N_{\Lambda_v}(\mathbf{x}) \neq 1\}$ is finite. Therefore, given an adelic norm $\mathcal{F}$ on V, it makes sense to set

$$H_{\mathcal{F}}(\mathbf{x}) = \prod_{v \in \mathcal{M}_K} N_v(\mathbf{x})^{d_v},$$

for all $\mathbf{0} \neq \mathbf{x} \in V$. We set by definition $H_{\mathcal{F}}(\mathbf{0}) = 1$. The function $H_{\mathcal{F}} : V \to \mathbb{R}$ so defined is called the *height* associated to $\mathcal{F}$. Note that the product formula implies that $H_{\mathcal{F}}$ is homogeneous, i.e. $H_{\mathcal{F}}(\lambda\mathbf{x}) = H_{\mathcal{F}}(\mathbf{x})$ for all $\lambda \in K^\times$.

Examples. (a) Let $V = K^n$. Set

$$\|\,\mathbf{x}\,\|_v = \begin{cases} \left(\sum_{i=1}^n |x_i|_v^2\right)^{\frac{1}{2}} & \text{if } v \in \mathcal{M}_K^\infty \\ \sup_{1 \leq i \leq n} |x_i|_v & \text{if } v \in \mathcal{M}_K^0. \end{cases}$$

Then $\mathcal{E} = \{\|\cdot\|_v\}_{v \in \mathcal{M}_K}$ is an adelic norm on K^n and its associated height function $H_{\mathcal{E}}$ is the ℓ^2 Northcott-Weil height. By changing the ℓ^2-norms at the archimedean

[1]Some comments on why we have chosen this definition of adelic norm will be provided in the remark following the definition of the operator height associated to $\mathcal{F}$.

places into either ℓ^1 or ℓ^∞-norms one recovers the other two Northcott-Weil heights that are commonly used. Note that $H_{\mathcal{E}}$ is invariant under field extensions.
(b) Let V be an n-dimensional K-vector space and $\underline{b}=\{\mathbf{y}_1,\dots,\mathbf{y}_n\}$ a basis of V over K. Let $\iota_{\underline{b}}$ be the isomorphism of V to K^n defined by mapping $\mathbf{y}_i$ to $\mathbf{e}_i$, where $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is the canonical basis of K^n. Set $\|\,\mathbf{x}\,\|_{b,v}=\|\,\iota_{\underline{b}}(\mathbf{x})\,\|_v$, then $\mathcal{F}_{\underline{b}}=\{\|\cdot\|_{b,v}, v\in\mathcal{M}_K\}$ is an adelic norm.
(c) Let $\mathcal{T}=(T_v)$ be an element of $\mathrm{GL}_n(K_{\mathbb{A}})$, the adele group of $\mathrm{GL}_n(K)$. Define $N_v:K_v^n\to\mathbb{R}$ by setting $N_v(\mathbf{x})=\|\,T_v(\mathbf{x})\,\|_v$. Then $\mathcal{F}_{\mathcal{T}}=\{N_v,\ v\in\mathcal{M}_K\}$ is an adelic norm on K^n. The associated height $H_{\mathcal{F}_{\mathcal{T}}}$ was used by D.Roy and J.Thunder in [6], where it is called the twisted height associated to $\mathcal{T}$.

Let us point out that the height arising from the adelic norms defined in the examples include all the height functions that are commonly used in the literature.

Let $\mathcal{F}$ be an adelic norm on V. The function $H_{\mathcal{F}}^{\mathrm{op}}:\mathrm{End}\big(V\big)\to\mathbb{R}$ defined by setting

$$H_{\mathcal{F}}^{\mathrm{op}}(T)=\sup_{\mathbf{x}\in V}\frac{H_{\mathcal{F}}\big(T(\mathbf{x})\big)}{H_{\mathcal{F}}(\mathbf{x})}$$

is called the *operator height* associated to $\mathcal{F}$ (or to $H_{\mathcal{F}}$). The following properties of $H_{\mathcal{F}}^{\mathrm{op}}$ are an immediate consequence of the above definition:

(1) $H_{\mathcal{F}}^{\mathrm{op}}(\lambda T)=H_{\mathcal{F}}^{\mathrm{op}}(T)$
(2) $H_{\mathcal{F}}^{\mathrm{op}}(TS)\le H_{\mathcal{F}}^{\mathrm{op}}(T)H_{\mathcal{F}}^{\mathrm{op}}(S)$.

Note that property (1) ensures us that $H_{\mathcal{F}}^{\mathrm{op}}$ descends to a well-defined function on $\mathbb{P}\big(\mathrm{End}(K^n)\big)$. We will see in the next section that $H_{\mathcal{F}}^{\mathrm{op}}$ is well defined meaning that the $H_{\mathcal{F}}^{\mathrm{op}}(T)<\infty$.

Remark. Condition (b) in the definition of adelic norm is in some sense a strong one. In fact it is immediate to verify that it is equivalent to require that there exists a basis $\underline{b}$ of V such that $N_v=\|\cdot\|_{b,v}$ for all but finitely many $v\notin\mathcal{M}_K^0$. A possible and natural way of relaxing condition (b) would be to require only that the set $\{v\in\mathcal{M}_K \mid N_v(\mathbf{x})\neq 1\}$ is finite for all $0\neq\mathbf{x}\in V$. This condition is indeed sufficient to have a well-defined height function on V attached to $\mathcal{F}$, but not for ensuring that the operator height is well defined, as shown by the following example. Consider the family of norms $\mathcal{F}=\{N_p \mid \mathcal{M}_{\mathbb{Q}}\}$, where $N_p:\mathbb{Q}_p^2\to\mathbb{R}$ is defined as

$$N_p(x_1,x_2)=\begin{cases}\max\{|x_1|_p,|px_2|_p\} & \text{if } p\neq\infty\\ \max\{|x_1|,|x_2|\} & \text{if } p=\infty.\end{cases}$$

Clearly N_p is an ultrametric norm for any $p\neq\infty$. Moreover given (x_1,x_2) there are only finitely many p's for which $N_p(x_1,x_2)\neq 1$. Let q be a prime

$$H_{\mathcal{F}}(q,1)=q\cdot N_q(q,1)=qq^{-1}=1, \tag{1.1}$$

while

$$H_{\mathcal{E}}(q,1)=\sqrt{q^2+1}.$$

It follows that

$$\frac{H_{\mathcal{E}}(q,1)}{H_{\mathcal{F}}(q,1)}=\sqrt{q^2+1}$$

which is clearly unbounded as $q \to \infty$. This fact contradicts the conclusion of lemma 2.1 (see next section). Therefore $\mathcal{F}$ cannot be an adelic norm. Let $T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. We want to show that

$$\sup_{\mathbf{x}\in\mathbb{Q}^2} \frac{H_{\mathcal{F}}\big(T(x)\big)}{H_{\mathcal{F}}(\mathbf{x})} = +\infty.$$

We will accomplish this by exhibiting a sequence $\{\mathbf{x}_n\}$ of vectors in $\mathbb{Q}^2$ such that $H_{\mathcal{F}}(\mathbf{x}_n) = 1$ and $H_{\mathcal{F}}\big(T(\mathbf{x}_n)\big) \to \infty$ as $n \to \infty$. First of all note that if $x = (2^m q, 1)$, with q being any integer, then $H_{\mathcal{F}}(x) \geq 2^{m-1}$. Then, let $\{p_n\}$ be a sequence of prime numbers such that $p_n \equiv 1 \bmod 2^n$. The existence of such a sequence is guaranteed by repeated applications of Dirichlet's theorem on primes in arithmetic progressions. Then let $\{\mathbf{x}_n = (p_n, 1)\}$. By (1.1) $H_{\mathcal{F}}(\mathbf{x}_n) = 1$, on the other hand $T(\mathbf{x}_n) = (p_n - 1, 1) = (2^n q_n, 1)$ and so $H_{\mathcal{F}}\big(T(x_n)\big) \geq 2^{n-1}$.

We conclude this section by giving the definition of the spectral height. Let us recall the definition of the local spectral radii. Let F be a complete local field and W a finite dimensional F-vector space. The *spectral radius* of $T\in \mathrm{End}\big(W\big)$ is $\rho_F(T) = \sup_{\lambda\in\, \mathrm{sp}(T)} |\lambda|_{F(\lambda)}$, where $\mathrm{sp}(T)$ is the set of characteristic roots of T, and $|\cdot|_{F(\lambda)}$ is the unique extension of $|\cdot|_F$ to $F(\lambda)$. Given $T\in \mathrm{End}\big(V\big)$ we set $\rho_v(T) = \rho_{K_v}(T_v)$, where T_v is the extension of T to V_v by K_v-linearity. If T is not nilpotent we set

$$H_s(T) = \prod_{v\in\mathcal{M}_K} \rho_v(T)^{d_v}.$$

We set $H_s(T) = 1$ for any nilpotent transformation. The function thereby defined is called the *spectral height* and enjoys the following properties:

(S1) $H_s(\lambda T) = H_s(T)$.
(S2) $H_s(T) \geq 1$.
(S3) $H_s(T^k) = H_s(T)^k$.
(S4) *H_s is invariant under conjugation.*
(S5) *If T_s is the semisimple, part of T then $H_s(T) = H_s(T_s)$.*
(S6) *If $T, T'\in \mathrm{End}\big(V\big)$ commute, $H_s(TT') \leq H_s(T)H_s(T')$.*
(S7) *H_s is invariant under field extension.*

Properties (S3)-(S6) are direct consequences of the behavior of the spectrum under the various operations considered (see [5]). Property (S1) follows from the product formula while (S7) is derived in a standard way from the formula for local degrees (see [5, ch.3 §1]). Finally (S2) follows from (S1) and (S7).

2. A reduction lemma

The main goal of this section is to show, that in order to establish theorems A and B in full generality, it is sufficient to prove them for H^{op} on K^n (recall that H^{op} is the operator height associated to the standard ℓ^2 adelic norm on K).

Lemma 2.1. *Let $\mathcal{F} = \{N_v, v\in\mathcal{M}_K\}$ be an adelic norm on V. Then there exists a constant $C > 0$ and an isomorphism $\iota : V \to K^n$ such that*

$$C^{-1} H^{\mathrm{op}}(T^\iota) \leq H^{op}_{\mathcal{F}}(T) \leq C H^{\mathrm{op}}(T^\iota),$$

where $T^\iota = \iota \circ T \circ \iota^{-1} \in \mathrm{End}(K^n)$.

Proof. Let Λ be a K-lattice such that $N_v = N_{\Lambda_v}$ for all but finitely many $v \in \mathcal{M}^0_K$. By definition Λ contains a basis $\underline{b} = \{\mathbf{y}_1, \dots, \mathbf{y}_n\}$ of V over K. It follows that there exists a finite set of absolute values $\mathcal{P}$, containing $\mathcal{M}^\infty_, K$, such that $N_v = \| \cdot \|_{b,v}$ for all $v \notin \mathcal{P}$. Let $\iota : V \to K^n$, $\mathbf{y}_i \mapsto \mathbf{e}_i$. If $v \notin \mathcal{S}$, then $N_v(\mathbf{x}) = \| \mathbf{x} \|_{b,v} = \| \iota(\mathbf{x}) \|_v$. On the other hand all the norms on a finite dimensional vector space over a complete field are equivalent, thus $\mathcal{P}$ being finite we find that there exists $C_1, C_2 > 0$, such that

$$C_1 \cdot H\big(\iota(\mathbf{x})\big) \leq H_{\mathcal{F}}(\mathbf{x}) \leq C_2 \cdot H\big(\iota(\mathbf{x})\big).$$

Then an easy computation shows

$$C^{-1} H^{\mathrm{op}}(T^\iota) \leq H^{\mathrm{op}}_{\mathcal{F}}(T) \leq C H^{\mathrm{op}}(T^\iota)$$

with $C = C_2/C_1 > 0$. ∎

Let $\mathcal{F}$ be a adelic norm on V. Given $B > 0$, set

$$\Omega\big(\mathbb{P}\big(\mathrm{End}(V)\big), H^{\mathrm{op}}_{\mathcal{F}}, B\big) = \big\{[T] \in \mathbb{P}\big(\mathrm{End}(V)\big) \;\big|\; \mathrm{rank}\,(T) \geq 2 \,\mathrm{and}\, H^{\mathrm{op}}_{\mathcal{F}}(T) \leq B\big\}.$$

The next lemma achieves the goal of this section.

Lemma 2.2. *If theorems A and B hold for H^{op} on K^n, then they hold for the operator height associated to any adelic norm on an n-dimensional K-vector space.*

Proof. Let $\iota : V \to K^n$ and $C > 0$ be as in the conclusion of the previous lemma. It follows that for every $B > 0$, the map $T \mapsto T^\iota$ gives rise to an injection $\Omega\big(\mathbb{P}\big(\mathrm{End}(V)\big), H^{\mathrm{op}}_{\mathcal{F}}, B\big) \hookrightarrow \Omega\big(\mathbb{P}\big(\mathrm{End}(K^n)\big), H^{\mathrm{op}}, BC\big)$.

Regarding the assertion of theorem A we first note that if theorem A holds for H^{op} on K^n, then

$$\limsup_{k\to\infty} H^{\mathrm{op}}_{\mathcal{F}}(T^k)^{\frac{1}{k}} \leq \limsup_{k\to\infty} \big(C H^{\mathrm{op}}((T^\iota)^k)\big)^{\frac{1}{k}} = \lim_{k\to\infty} \big(H^{\mathrm{op}}((T^\iota)^k)\big)^{\frac{1}{k}} = H_s(T^\iota)$$

and

$$\liminf_{k\to\infty} \big(C^{-1} H^{\mathrm{op}}(T^{\iota k})\big)^{\frac{1}{k}} \leq \liminf_{k\to\infty} H^{\mathrm{op}}_{\mathcal{F}}(T^k)^{\frac{1}{k}} = \lim_{k\to\infty} \big(H^{\mathrm{op}}((T^\iota)^k)\big)^{\frac{1}{k}} = H_s(T^\iota).$$

Combining these inequalities with the fact that $H_s(T) = H_s(T^\iota)$ yields the desired limit formula for $H^{\mathrm{op}}_{\mathcal{F}}$. ∎

Thanks to lemma 2.2. we can restrict our attention to H on K^n. Let us introduce the heights, related to H, that will be useful in the sequel. If $\mathcal{X}$ is one-dimensional we set $H(\mathcal{X}) = H(\mathbf{x})$, where $\mathbf{0} \neq \mathbf{x} \in \mathcal{X}$ is any non-zero element. A subspace $\mathcal{X} \subset K^n$ of dimension ℓ determines a one-dimensional subspace $\mathcal{P}_\ell(\mathcal{X})$ of K^m, where $m = \binom{n}{\ell}$ (via the Plücker map), allowing us to set $H(\mathcal{X}) = H\big(\mathcal{P}_\ell(\mathcal{X})\big)$, cf. [7] where this height was firstly introduced.

Next we want to introduce an auxiliary height function on $\mathrm{End}\big(K^n\big)$, which is defined as the product of local operator norms. More precisely the norm $\| \cdot \|_v$ induces a norm on $\mathrm{End}\big((K_v)^n\big)$, defined by

$$\| S \|_v = \sup_{\mathbf{x} \in (K_v)^n - \{\mathbf{0}\}} \frac{\| S(\mathbf{x}) \|_v}{\| \mathbf{x} \|_v}.$$

Explicitly, given $S = (s_{ij}) \in M_n(K_v)$, we have $\| S \|_v = \sup_{1\le i,j\le n} |s_{ij}|_v$ if $v \in \mathcal{M}_K^0$. If $v \in \mathcal{M}_K^\infty$, then $\| T \|_v = \sup_{\lambda \in \mathrm{sp}(T^*T)} \sqrt{\lambda}$ where T^* is the adjoint of T. It follows that for $0 \neq T \in \mathrm{End}\big(K^n\big)$, we have $\| T \|_v = 1$ for all but finitely many v's. Therefore it makes sense to set:

$$H(T) = \prod_{v \in \mathcal{M}_K} \| T \|_v^{d_v}.$$

As usual, we set $H(0) = 1$. The elementary properties of $H : \mathrm{End}\big(K^n\big) \to \mathbb{R}$ are:

(H1) $H(\lambda T) = H(T)$;
(H2) $H(T) \geq 1$;
(H3) $H(TT') \leq H(T) \cdot H(T')$;
(H4) $H^{\mathrm{op}}(T) \leq H(T)$
(H5) *Let* $\phi : \mathrm{End}(K^n) \to K^{n^2}$ *be an isomorphism which assigns to* T *the* n^2*-tuple formed by its entries (ordered in some way). Then there exists a constant* $C > 0$ *such that:* $H(T) \leq H\big(\phi(T)\big) \leq CH(T)$ *for all* $T \in \mathrm{End}(K^n)$.

Property (H4) follows at once from the definition of $H^{\mathrm{op}}(T)$ and $H(T)$. Of the remaining properties, the only one which is not a straightforward consequence of the corresponding properties of the local norms and the product formula is the inequality $H\big(\phi(T)\big) \leq CH(T)$, which is proven exactly as in lemma 2.1.

Remark. Note that H4 and lemma 2.1 implies that $H_{\mathcal{F}}^{\mathrm{op}}(T) < \infty$ for all $T \in \mathrm{End}\big(V\big)$, V being a finite dimensional K-vector space and $\mathcal{F}$ any adelic norm on it.

3. Comparison Results for Heights on $\mathrm{End}(K^n)$

The main result of this section is theorem 3.3., which establishes a comparison result between H and H^{op}, which, beside of being interesting in its own right, will be used in the proof of theorem A. In fact our proof of theorem A consists of two steps: first we prove the desired limit formula with H in place of H^{op}, then we show, by means of theorem 3.3, that this forces the limit formula to hold also for H^{op}.

Let $\mathcal{X} \subseteq K^n$ be a subspace. For each $v \in \mathcal{M}_K$ let $\mathcal{X}_v$ be the closure of $\mathcal{X}$ in K_v^n. Define $\| \cdot \|_{\mathcal{X}_v}$, the *seminorm relative to* $\mathcal{X}_v$, to be

$$\| \mathbf{y} \|_{\mathcal{X}_v} = \inf_{\mathbf{x} \in \mathcal{X}_v} \| \mathbf{y} - \mathbf{x} \|_v.$$

The global function associated to the local seminorms is

$$\begin{aligned} d_{\mathcal{X}} : K^n \setminus \mathcal{X} &\longrightarrow \mathbb{R} \\ \mathbf{y} \mapsto d_{\mathcal{X}}(\mathbf{y}) &= \prod_{v \in \mathcal{M}_K} \|\mathbf{y}\|_{\mathcal{X}_v}^{d_v}. \end{aligned}$$

The significance of $d_{\mathcal{X}}$ is explained by the following proposition

Proposition 3.1. *Suppose* $T \in \mathrm{End}(K^n)$ *is not zero. Let* $\mathcal{X} = \ker T$. *Then*

$$H(T) = \sup_{\mathbf{y} \in K^n \setminus \mathcal{X}} \frac{H\big(T(\mathbf{y})\big)}{d_{\mathcal{X}}(\mathbf{y})}. \tag{3.1}$$

Proof.

$$\| T \|_v = \sup_{\mathbf{z}\in K_v^n\setminus\mathcal{X}_v} \{ \sup_{\mathbf{x}\in\mathcal{X}_v} \frac{\| T(\mathbf{z}) \|_v}{\| \mathbf{z}-\mathbf{x} \|_v} \} = \sup_{\mathbf{z}\in K_v^n\setminus\mathcal{X}_v} \{ \frac{\| T(\mathbf{z}) \|_v}{\inf\limits_{\mathbf{x}\in\mathcal{X}_v} \| \mathbf{z}-\mathbf{x} \|_v} \} = \sup_{\mathbf{z}\in K_v^n\setminus\mathcal{X}_v} \frac{\| T(\mathbf{z}) \|_v}{\| \mathbf{z} \|_{\mathcal{X}_v}},$$

hence

$$\sup_{\mathbf{y}\in K^n\setminus\mathcal{X}} \frac{H\big(T(\mathbf{y})\big)}{d_{\mathcal{X}}(\mathbf{y})} \leq \prod_{v\in\mathcal{M}_K} \sup_{\mathbf{z}\in K_v^n\setminus\mathcal{X}_v} \frac{\| T(\mathbf{z}) \|_v^{d_v}}{\| \mathbf{z} \|_{\mathcal{X}_v}^{d_v}} = H(T).$$

To prove the reverse inequality, let $\mathcal{P} \supset \mathcal{M}_K^\infty$ be a finite set of places such that

$$\text{(3.2)}\quad \| T \|_v = 1 \qquad \forall v \notin \mathcal{P} \qquad \text{and} \qquad \frac{H\big(T(\mathbf{z})\big)}{d_{\mathcal{X}}(\mathbf{z})} = \prod_{v\in\mathcal{P}} \frac{\| T(\mathbf{z}) \|_v^{d_v}}{\| \mathbf{z} \|_{\mathcal{X}_v}^{d_v}} \qquad \forall \mathbf{z}\in K^n\setminus\mathcal{X}.$$

The existence of $\mathcal{P}$ is guaranteed by lemma 3.2 below. Given $\epsilon > 0$ let $\delta > 0$ be such that $\prod\limits_{v\in\mathcal{P}} \| T \|_v^{d_v} \leq \epsilon + \prod\limits_{v\in\mathcal{P}} \big(\| T \|_v^{d_v} - \delta \big)$. By the weak approximation theorem we can find $\mathbf{z}\in K^n$ such that

$$\| T \|_v^{d_v} - \delta \leq \frac{\| T(\mathbf{z}) \|_v^{d_v}}{\| \mathbf{z} \|_v^{d_v}}$$

for all $v\in\mathcal{P}$. Taking the product over $\mathcal{P}$ and using the equalities (3.2) yields

$$H(T) = \prod_{v\in\mathcal{P}} \| T \|_v^{d_v} \leq \epsilon + \prod_{v\in\mathcal{P}} \big(\| T \|_v^{d_v} - \delta \big) \leq \epsilon + \prod_{v\in\mathcal{P}} \frac{\| T(\mathbf{z}) \|_v^{d_v}}{\| \mathbf{z} \|_v^{d_v}} \leq \epsilon + \frac{H\big(T(\mathbf{z})\big)}{d_{\mathcal{X}}(\mathbf{z})}$$

completing the proof of the lemma. ■

Lemma 3.2. *Let $0 \neq T \in \mathrm{M}_n\big(K\big)$ and set $\mathcal{X} = \ker T$. Then there exists a finite set of places $\mathcal{P} \supset \mathcal{M}_K^\infty$*

(a) $\| T \|_v = 1 \qquad \forall v \notin \mathcal{P}$

(b) $\dfrac{H\big(T(\mathbf{z})\big)}{d_{\mathcal{X}}(\mathbf{z})} = \prod\limits_{v\in\mathcal{P}} \dfrac{\| T(\mathbf{z}) \|_v^{d_v}}{\| \mathbf{z} \|_{\mathcal{X}_v}^{d_v}}, \qquad \forall \mathbf{z}\in K^n \setminus \mathcal{X}.$

Proof. If $\mathrm{rank}\,(T) = n$ the lemma reduces to the fact that T belongs to $\mathrm{GL}_n\big(\mathcal{O}_v\big)$ for all but finitely many $v\in\mathcal{M}_K$. So we can assume $r < n$. To prove the lemma it suffices to exhibit a subspace $\mathcal{Y} \subset K^n$ of dimension $r = \mathrm{rank}\,(T)$ and a finite set of places $\mathcal{P} \supset \mathcal{M}_K^\infty$ such that for all $v \notin \mathcal{P}$ we have :

(a′) $\| \mathbf{y} \|_{\mathcal{X}_v} = \| \mathbf{y} \|_v \qquad \forall \mathbf{y}\in\mathcal{Y}_v$.

(b′) $\| T(\mathbf{y}) \|_v = \| \mathbf{y} \|_{\mathcal{X}_v} \qquad \forall \mathbf{y}\in\mathcal{Y}_v$.

Suppose first that $\mathcal{X}$ is spanned by the last $n - r$ elements of the canonical basis of K^n. In this case we take $\mathcal{Y}$ to be the span of the first r elements of the canonical basis of K^n, and properties (a′) and (b′) are trivially verified. In general we proceed as follows: choose $S\in\mathrm{M}_n\big(K\big)$, S invertible, such that $\mathcal{X}$ is spanned by $S(\mathbf{e}_{r+1}), \dots, S(\mathbf{e}_n)$. Now note that (1) and (2) hold for $T\circ S$ and that S is an isometry with respect to $\| \cdot \|_v$ for all but finitely many $v\in\mathcal{M}_K$. Let $\mathcal{Y} = S(< \mathbf{e}_1, \dots, \mathbf{e}_r >)$ and $\mathcal{P}$ be the finite set formed by those places of K for which either S is not a $\| \cdot \|_v$-isometry, or one of conditions (a′) and (b′) does not hold for $T\circ S$. It is straightforward to verify that $\mathcal{Y}$ and $\mathcal{P}$ satisfies (a′) and (b′).
■

Theorem 3.3. *Let* $T \in \mathrm{End}\big(K^n\big)$*, and set* $\mathcal{X} = \ker T$*. Then*

(a) *If* T *is invertible, then* $H^{\mathrm{op}}(T) = H(T)$.

(b) *If* $1 < \mathrm{rank}\,(T) < n$*, then there a constant* $C(K,n) \geq 1$ *depending only on* K *and* n.

$$H^{\mathrm{op}}(T) \geq H(T)C(K,n)^{-1}H(\mathcal{X})^{-1}.$$

(c) *If* $\mathrm{rank}\,(T) = 1$*, then* $H^{\mathrm{op}}(T) = H(T)H(\mathcal{X})^{-1}$.

The proof of theorem 3.3 is based upon the following lemma:

Lemma 3.4. (a) *Let* $\mathcal{X} \subset K^n$ *be a subspace such that* $1 \leq \dim_K \mathcal{X} < n-1$*. Then*

$$\inf_{\mathbf{x}\in\mathcal{X}} H(\mathbf{y}-\mathbf{x}) \leq C(K,n)d_{\mathcal{X}}(\mathbf{y})H(\mathcal{X})$$

for all $\mathbf{y} \notin \mathcal{X}$.

(b) *If* $\dim_K \mathcal{X} = n-1$*, then* $H(\mathcal{X}) = d_{\mathcal{X}}(\mathbf{y})^{-1}$.

Proof. It follows from lemma 4 of [12] that $H(\mathcal{X})d_{\mathcal{X}}(\mathbf{y}) = H\big(\langle \mathcal{X}, \mathbf{y}\rangle\big)$. If $\dim_K \mathcal{X} = n-1$, then $\langle \mathcal{X}, \mathbf{y}\rangle = K^n$, proving (b) (recall that $H(K^n) = 1$). Now assume $\dim_K \mathcal{X} < n-1$. By applying the ℓ^2-version of Siegel's lemma (see [11]), we find a a basis $\{\mathbf{z}_1, \dots, \mathbf{z}_{\ell+1}\}$ of $\langle \mathcal{X}, \mathbf{y}\rangle$ and a constant $C(n,K)$ depending on n and K, but not on $\mathcal{X}$, such that

$$\prod_{i=1}^{\ell+1} H(\mathbf{z}_i) \leq C(n,K)H\big(\langle \mathcal{X}, \mathbf{y}\rangle\big) = C(n,K)H(\mathcal{X})d_{\mathcal{X}}(\mathbf{y})$$

Now the height of any $\mathbf{y} \in K^n$ is at least one and at least one of the $\mathbf{z}_i$'s has to be of the form $\mathbf{z}_i = \mathbf{y} - \mathbf{x}_i$, completing the proof of the proposition. ∎

Proof of theorem 3.3.
(a) If $\mathrm{rank}\,(T) = n$ then $d_{\mathcal{X}} = H$ and so (a) was proven in proposition 3.1
(b) Let $\mathcal{X} = \ker T$ and $C = C(n,K)$. By proposition 3.1 and lemma 3.4., we have

$$H(T) = \sup_{\mathbf{y}\notin\mathcal{X}} \frac{H\big(T(\mathbf{y})\big)}{d_{\mathcal{X}}(\mathbf{y})} \leq C \cdot H(\mathcal{X}) \cdot \left\{ \sup_{\mathbf{y}\in K^n} \frac{H\big(T(\mathbf{y})\big)}{\inf_{\mathbf{x}\in\mathcal{X}} H(\mathbf{y}-\mathbf{x})} \right\} = C \cdot H(\mathcal{X})H^{\mathrm{op}}(T).$$

(c) Let $\mathcal{Y}$ be the image of T. Then $H^{\mathrm{op}}(T) = H(\mathcal{Y}) = H(T(\mathbf{y}))$ for any $\mathbf{y} \notin \mathcal{X}$. By proposition 3.1 $H(T) = H(T(\mathbf{y}))d_{\mathcal{X}}(\mathbf{y})^{-1}$, hence (c) follows from lemma 3.4. ∎

4. Proof of Theorems A and B

The main ingredients for the proof of theorem A, besides theorem 3.3, are the local Gelfand-Beurling formulae, which we recall below:

Theorem 4.1. *Let* K *be a number field and* v *a place of* K*. Suppose* S *belongs to* $\mathrm{End}\big(K_v^n\big)$*. Then*

$$\lim_{k\to\infty} \|S^k\|_v^{\frac{1}{k}} = \rho_v(S). \tag{4.1}$$

Proof. If $v\in\mathcal{M}_K^\infty$, then (4.1) is a special case of the Gelfand-Beurling formula for Banach algebras. For a simultaneous proof of the real and complex case see [3]. If $v\in\mathcal{M}_K^\infty$, then (4.1) is proven (in a more general setting) in [1, theorem 7.2.1.] A direct and elementary proof is in [10, supplement 3, theorem 14].

The next theorem completes the proof of theorem A of the introduction.

Theorem 4.2. *Let $T\in\mathrm{End}(K^n)$. Then*

$$H_s(T) = \lim_{k\to\infty} \big(H^{\mathrm{op}}(T^k)\big)^{\frac{1}{k}}.$$

Proof. We will first prove that

$$H_s(T) = \lim_{k\to\infty} \big(H(T^k)\big)^{\frac{1}{k}}. \tag{5.2}$$

Then we will deduce the analogous formula for H^{op} with the aid of theorem 3.3. Fix $T\in\mathrm{End}(K^n)$ and let $\mathcal{P}\subset\mathcal{M}_K$ be defined by requiring that $v\in\mathcal{P}$ if and only if either $\rho_v(T)\neq 1$ or $\|T^k\|_v\neq 1$ for some $k\geq 1$. Now the first condition is clearly verified only for finitely many v's. By lemma 3.2 the second condition is also verified only by finitely many v's. It follows that $\mathcal{P}$ is a finite set. Moreover:

$$H_s(T) = \prod_{v\in\mathcal{P}} \rho_v(T_v)^{d_v}; \qquad H(T^k) = \prod_{v\in\mathcal{P}} \| T_v^k \|_v^{d_v} \qquad \text{for all } k\geq 1.$$

Since $\mathcal{P}$ is finite we can exchange the product with the limit, formula (4.2) follows from the local Gelfand-Beurling formulae. In particular, this yields the theorem for T invertible because in this case $H(T) = H^{\mathrm{op}}(T)$ by theorem 3.3.(a). Suppose now that T is singular. Note that $\ker T^k = \ker T^h$ for all $h,k\geq n$. Let $B = H(\ker T^n)$. Then, by theorem 3.3.(b) and (c), we have

$$\liminf_{k\to\infty} H^{\mathrm{op}}(T^k)^{\frac{1}{k}} \geq \liminf_{k\to\infty} \left(\frac{1}{C_K B}\right)^{\frac{1}{k}} H(T^k)^{\frac{1}{k}} \geq \liminf_{k\to\infty} H(T^k)^{\frac{1}{k}} = H_s(T).$$

On the other hand $H^{\mathrm{op}}(T)\leq H(T)$, so

$$\limsup_{k\to\infty} H^{\mathrm{op}}(T^k)^{\frac{1}{k}} \leq \limsup_{k\to\infty} H(T^k)^{\frac{1}{k}} = H_s(T). \quad \blacksquare$$

Next we deal with Northcott's finiteness theorem. As was previously mentioned, it is indeed possible to have an infinite family of rank-one maps which are pairwise not homothetic and which all have height bounded by the same constant. Consider for example the family $\{[T_n], n\in\mathbb{Z}\}$, where $T_n = \left(\begin{smallmatrix} 1 & n \\ 1 & n \end{smallmatrix}\right)$. Then $H^{\mathrm{op}}(T_n) = \sqrt{2}$ for all n, and T_n is not homothetic to T_m, if $n\neq m$. Let

$$\Omega_1^C\big(\mathbb{P}\big(\mathrm{End}(K^n)\big), B\big) = \big\{[T] \;\big|\; \mathrm{rank}\,(T) = 1, H^{\mathrm{op}}(T)\leq B \text{ and } H\big(\ker T\big)\leq C\big\}\,.$$

Then our finiteness results can be formally stated as follows:

Theorem 4.3. *Let $B \geq 1$. Then*
(a) *The set $\Omega\big(\mathbb{P}\big(\mathrm{End}(K^n)\big), H^{\mathrm{op}}, B\big)$ is finite.*
(b) *The set $\Omega_1^C\big(\mathbb{P}\big(\mathrm{End}(K^n)\big), B\big)$ is finite for all $C \geq 1$.*

Proof. (a) Northcott's finiteness theorem for projective space implies that we can choose $\mathbf{0} = \mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m \in K^n$, in such a way that any $\mathbf{y} \in K^n$ with $H(\mathbf{y}) \leq B$ is a non-zero scalar multiple of one and only one of the $\mathbf{x}_i$'s. Suppose $T = (\mathbf{t}_1 \ \dots \ \mathbf{t}_n)$ is such that $H^{\mathrm{op}}(T) \leq B$. Then $\mathbf{t}_j = \lambda_j(T)\mathbf{x}_{i_j^T}$, with $\lambda_j \neq 0$ and $i_j^T \in \{0, \dots, m\}$. The n-tuple $(i_1^T, \dots, i_n^T)$ depends only on the image of T in $\mathbb{P}\big(\mathrm{End}(K^n)\big)$. To prove (a) it suffices to show that there are only finitely many elements of $\Omega\big(\mathbb{P}\big(\mathrm{End}(K^n)\big), B\big)$ having the same associated n-tuple. Fix an n-tuple $(i_1, \dots, i_n)$ which arises as associated to some element $[T] = [(\mathbf{t}_1 \ \dots \ \mathbf{t}_n)] \in \Omega\big(\mathbb{P}\big(\mathrm{End}(K^n)\big), B\big)$. Since rank $(T) \geq 2$, there exists h, k such that $\mathbf{0} \neq \mathbf{x}_{i_h} \neq \mathbf{x}_{i_k} \neq \mathbf{0}$, and

$$H(\mathbf{x}_{i_h} + \lambda_h(T)^{-1}\lambda_j(T)\mathbf{x}_{i_j}) \leq \sqrt{2}B; \quad H(\mathbf{x}_{i_k} + \lambda_k(T)^{-1}\lambda_j(T)\mathbf{x}_{i_j}) \leq \sqrt{2}B \tag{4.3}$$

for all j's for which $\mathbf{x}_{i_j} \neq \mathbf{0}$. Northcott's Theorem for projective spaces implies that given $\mathbf{0} \neq \mathbf{y}$ and $\mathbf{z}$ linearly independent, there are only finitely many values of $\lambda \in K^\times$ for which vectors of the form $\mathbf{y} + \lambda\mathbf{z}$ have bounded height. Combining this with the inequalities (4.3), we find that the ratios $\lambda_h(T)^{-1}\lambda_j(T)$ can assume only finitely many values. Hence $(i_1, \dots, i_n)$ is associated only to finitely many $[T] \in \Omega\big(\mathbb{P}\big(\mathrm{End}(K^n)\big), B\big)$.
(b) Theorem 3.3. implies that

$$\Omega_1^C\big(\mathbb{P}\big(\mathrm{End}(K^n)\big), B\big) \subset \big\{T \in \mathbb{P}\big(\mathrm{End}(K^n)\big) \ \big| \ H(T) \leq B \cdot C\big\}.$$

But the set on the right is a finite set by (H5) and Northcott's finiteness theorem for projective spaces. ■

References

1. V. G. Berkovich, *Spectral Theory and Analytic Geometry Over Non-archimedean Fields*, Mathematical Surveys and Monographs 33, AMS, Providence R.I., 1990.
2. E. B. Burger and J. D. Vaaler, *On the decomposition of vectors over number fields*, J. reine. angew. Math. **435** (1993), 197-219.
3. S. H. Kulkarni and B.V. Limaye, *Real Function Algebras*, Marcel Dekker, 1992.
4. S. Lang, *Fundamentals of Diophantine Geometry*, Springer-Verlag, 1983.
5. C. C. Mac Duffee, *The Theory of Matrices*, Ergebnisse der Mathematik und ihrer Grenzgebiete Band 2, Heft 5, Verlag von Julius Springer, 1933.
6. D. Roy and J.L. Thunder, *An Absolute Siegel's Lemma*, J. reine angew. Math **476** (1996), 1-26.
7. W. M Schmidt, *On heights of algebraic subspaces and diophantine approximations*, Ann. Math. **85** (1967), 430-472.
8. J. P. Serre, *Lectures on the Mordell-Weil theorem*, translated and edited by M. Brown from notes by M. Waldschmidt, Vieweg, Braunschweig, 1989.
9. V. Talamanca, *An Introduction to the Theory of Height Functions.*, Proceeding of "Primo Incontro Italiano di Teoria dei Numeri", Rend. Sem. Mat. Univ. Politec. Torino **53 N. 4** (1995), 217-234.
10. V. Talamanca, *Height Preserving Linear Transformations on Linear Spaces*, Ph. D. thesis, Brandeis University, 1995.

11. J. L. Thunder, *An adelic Minkowski-Hlawka theorem and an application to Siegel's lemma*, J. reine angew. Math. **475** (1996), 167-185.
12. J. D. Vaaler, *Small zeros of quadratic forms over number fields*, Trans. AMS **302** (1987), 281-296.
13. A. Weil, *Basic Number Theory*, Springer Verlag, 1967.
14. S. Zhang, *Small points and adelic metrics*, J. Algebraic Geometry **4** (1995), 281-300.

KdV Instituut voor Wiskunde, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands

E-mail address: valerio@wins.uva.nl